SELF-SIMILAR SOLUTIONS OF NONSTATIONARY EQUATIONS OF A PLANE LAMINAR MAGNETOHYDRODYNAMIC BOUNDARY LAYER

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Self-similar solutions of nonstationary equations of the boundary layer in ordinary hydrodynamics are discussed in [1, 2]. In this paper self-similar solutions of nonstationary equations of a plane magnetohydrodynamic boundary layer are sought. In this case, a transformation to curvilinear coordinates of a certain special form is employed. Its choice is determined by the requirements essential to reducing the equations of the boundary layer to a system of ordinary equations. H. Weyl's iterative method is used to solve the equations describing the flow over a plate suddenly set in motion.

1. Let us consider the transformation

$$\xi = x, \qquad \eta = \int_0^y \frac{dy}{w(x,t)}, \qquad t = t, \tag{1.1}$$

where w is a still arbitrary function. We shall obtain an approximation of the boundary layer in these coordinates if we write the equation of magnetohydrodynamics in them [3], and we shall seek their solution in the form [4]

$$x = x^*, \qquad y = \varepsilon y^*$$

$$v_1 = v_1^*, \quad v_2 = \varepsilon v_2^* \qquad (\varepsilon = R^{-1/2} \ll 1)$$

with accuracy to ε^2. In this approximation for the base vectors a_i, and the metric tensors g_{ik}, g^{ik} we obtain [5]

$$\mathbf{a}_1 = \mathbf{e}_1 + \frac{\partial y}{\partial \xi}\mathbf{e}_2, \quad \mathbf{a}_2 = w\mathbf{e}_2,$$

$$g_{ik} = \begin{pmatrix} 1 & w\,\partial y/\partial\xi \\ w\,\partial y/\partial\xi & w^2 \end{pmatrix}, \quad g^{ik} = \begin{pmatrix} 1 & 0 \\ 0 & w^{-2} \end{pmatrix}$$

(e_i are the unit vectors of the Cartesian axes). The relative change in an element of volume (Jacobian) is given by the formula

$$\sqrt{\det g_{ik}} = \sqrt{g} = w.$$

The equations of the boundary layer in the coordinates ξ, η for $R_m \approx R$ and a given external magnetic field $H_e\{H_e,\ 0,\ 0\}$ have the form

$$\frac{\partial v^1}{\partial t} + v^1\frac{\partial v^1}{\partial \xi} + v^2\frac{\partial v^1}{\partial \eta} =$$

$$= \left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial \xi}\right) + \frac{\nu}{\sqrt{g}}\frac{\partial}{\partial \eta}\left(\sqrt{g}\,g^{22}\frac{\partial v^1}{\partial \eta}\right) +$$

$$+ \frac{1}{4\pi\rho}\left(H^1\frac{\partial H^1}{\partial \xi} + H^2\frac{\partial H^1}{\partial \eta}\right) - \frac{1}{4\pi\rho}\left(H_e^1\frac{\partial H_e^1}{\partial \xi}\right),$$

$$\frac{\partial H^1}{\partial t} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \eta}\left\{\frac{1}{\sqrt{g}}(v_1H_2 - v_2H_1) + \frac{1}{\sqrt{g}}\frac{c^2}{4\pi\sigma}\frac{\partial H_1}{\partial \eta}\right\},$$

$$\frac{\partial}{\partial \xi}(\sqrt{g}\,v^1) + \frac{\partial}{\partial \eta}(\sqrt{g}\,v^2) = 0,$$

$$\frac{\partial}{\partial \xi}(\sqrt{g}\,H^1) + \frac{\partial}{\partial \eta}(\sqrt{g}\,H^2) = 0,$$

$$\frac{\partial T}{\partial t} + v^1\frac{\partial T}{\partial \xi} + v^2\frac{\partial T}{\partial \eta} =$$

$$= \frac{\nu}{\sqrt{g}}\frac{\partial}{\partial \eta}\left(\frac{\sqrt{g}}{P}g^{22}\frac{\partial T}{\partial \eta}\right) +$$

$$+ \nu g^{22}\left(\frac{\partial v^1}{\partial \eta}\right)^2 + \frac{c^2}{16\pi^2\sigma\nu\rho}\left(\frac{\partial H_1}{\partial \eta}\right)^2. \tag{1.2}$$

Here the superscripts are used to denote the contravariant components of the vectors and the subscripts the covariant components. In this case

$$v^1 = v_1 = v_x, \qquad v_2 = wv_y, \qquad H_1 = H^1 = H_x.$$

We select the following as boundary and initial conditions:

$$v_1 \to U(x,t), \quad H_1 \to H_e(x,t),$$

$$T \to T_e(x,t) \text{ when } \eta \to \infty,\ t > 0, \tag{1.3}$$

$$v_1 = v_2 = H_1 = H_2 = 0, \quad T = T_s(x,t) \text{ when } \eta = 0,\ t > 0,$$

$$v_1 = U(x,0), \quad H_1 = H_e(x,0), \quad T = T_e(x,0) \text{ when } t = 0.$$

The continuity equations are identically satisfied by introducing the stream function for the velocity Φ and the magnetic field Ψ:

$$H^1 = \frac{1}{w}\frac{\partial \Psi}{\partial \eta}, \quad H^2 = -\frac{1}{w}\frac{\partial \Psi}{\partial \xi},$$

$$v^1 = \frac{1}{w}\frac{\partial \Phi}{\partial \eta}, \quad v^2 = -\frac{1}{w}\frac{\partial \Phi}{\partial \xi}. \tag{1.4}$$

We shall seek the solution of Eqs. (1.2) in the form

$$\Psi = a(\xi,t)G(\eta),$$

$$\Phi = \varphi(\xi,t)f(\eta), \qquad T = T_e(\xi,t)\Theta(\eta). \tag{1.5}$$

From (1.4) and (1.5) we get

$$H^1 = H_e(\xi,t)\,G'(\eta),$$

$$H^2 = -\frac{1}{w}\left[H_e\frac{\partial w}{\partial \xi} + w\frac{\partial H_e}{\partial \xi}\right]G, \quad H_e = \frac{a}{w},$$

$$v^1 = U(\xi,t)\,f'(\eta),$$

$$v^2 = -\frac{1}{w}\left[U\frac{\partial w}{\partial \xi} + w\frac{\partial U}{\partial \xi}\right]f, \qquad U = \frac{\varphi}{w}.$$

In this case, Eqs. (1.2) become

$$-c_1(1-f')-c_2(1-f'^2)-c_3f''\eta-(c_2+c_4)ff''=$$
$$=f'''+S^2\{-c_2(1-G'^2)-(c_2+c_4)GG''\}=0,$$
$$c_1G'-c_3G''\eta=(c_2+c_4)[G'f-Gf']+\lambda G''',$$
$$2c_1\Theta-c_3\Theta\eta'+2c_2f'\Theta-(c_2+c_4)f\Theta'=$$
$$=P^{-1}\Theta''+M^2(f''^2+S^2\lambda G''^2). \qquad (1.6)$$

The primes indicate differentiation in respect to η,

$$c_1=\frac{w^2}{\nu U}\frac{\partial U}{\partial t},\quad c_2=\frac{w^2}{\nu}\frac{\partial U}{\partial\xi},\quad S^2=\frac{H_e^2}{4\pi\rho U^2},\quad \lambda=\frac{c^2}{4\pi\sigma\nu},$$
$$c_3=\frac{w}{\nu}\frac{\partial w}{\partial t},\quad c_4=\frac{wU}{\nu}\frac{\partial w}{\partial\xi},\quad M^2=\frac{U^2}{c_pT_e}. \qquad (1.7)$$

If we require that c_i, S, M should be constants, then Eqs. (1.6) become ordinary equations, and (1.7) will give the system for determining U, w, H_e, and T_e. Different cases of self-similarity are obtained by writing different combinations of c_i, and are possible in the following cases.

1°. Let us consider the stationary flows $c_1 = c_3 = 0$. The equations of motion will be of the form

$$f'''+ff''=\beta[f'^2-1+S^2(1-G'^2)]+S^2GG'',$$
$$\lambda G'''+fG''-f''G=0,$$
$$P^{-1}\Theta''+f\Theta'-2\beta\Theta f'+M^2(f''^2+S^2\lambda G''^2)=0$$
$$(c_4/c_2+1=1/\beta,\ c_2=\beta).$$

These are well-known equations [6] corresponding to a velocity of the external flow of the form $U\sim x^m$, $U\sim e^{\alpha x}$. In [7] the system corresponding to flow over a plate ($\beta = 0$) was solved by an iterative method. Analytic expressions were obtained for the first iterations for f_1, G_1.

2°. Let us consider nonstationary flows.

a) In the case $c_2 = c_4 = 0$, $c_3 \neq 0$, $c_1 \neq 0$ we have

$$w^2=2c_3t,\quad U\sim t^{1/2c_1/c_3},\quad H_e\sim t^{1/2c_1/c_3},\quad T_e\sim t^{c_1/c_3}.$$

To determine f, G, and Θ we obtain the equations

$$f'''+c_3f''\eta+c_1(1-f')=0,$$
$$\lambda G'''-c_1G'+c_3G''\eta=0,$$
$$P^{-1}\Theta''+c_3\Theta'\eta-2c_1\Theta+M^2(f''^2+S^2\lambda G''^2)=0. \qquad (1.8)$$

The first equation of (1.8), when S = 0, describes the initial flow of an ordinary fluid over an infinite plate suddenly set in motion at t = 0 and, in the first approximation, over a cylinder. For the cases $2c_3/c_1 \equiv \alpha = 0.1$ its solution was considered by Blasius [8], and for $\alpha = 1, 2, 3, 4$ by Hertler [8]. Watson [9] gave a solution for arbitrary α. System (1.8) describes an analogous flow of a conducting fluid in a homogeneous magnetic field directed parallel to the flow. The solutions of the first two equations of system (1.8) satisfying the conditions

$$f'\to1,\quad G'\to1,\quad \Theta\to1 \text{ when } \eta\to\infty,$$
$$f=f'=G=G'=0,\quad \Theta=\text{const when } \eta=0, \qquad (1.9)$$

are of the form

$$f(\eta_1)=\eta_1-2^{2\alpha}\Gamma(\alpha+1)\,g_{\alpha+1/2}(\eta_1)-\frac{1}{2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)},$$
$$G(\eta_1)=2^{2\alpha}\Gamma(\alpha+1)\,g_{\alpha+1/2}(\eta_1)-\frac{1}{2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}\qquad\left(\eta_1=\frac{\sqrt{2c_3}}{2}\eta\right),$$
$$g_\alpha(\eta)=\frac{2}{\sqrt{\pi}\,\Gamma(\alpha+1)}\int_\eta^\infty(\gamma-\eta)^{2\alpha}e^{-\gamma^2}d\gamma$$

($\Gamma(\alpha)$—is the gamma function).

b) In the case $c_2 = c_4 = c_3 = 0$, we have

$$U\sim\exp\frac{c_1t}{w^2},\quad H_e\sim\exp\frac{c_1t}{w^2},\quad w=\text{const},$$
$$f'''-c_1f'+c_1=0,\ \lambda G'''-c_1G'=0,$$
$$P^{-1}\Theta''-2c_1\Theta+M^2(f''+S^2\lambda G''^2)=0.$$

These equations describe a flow along an infinite flat plate moving at an exponentially increasing velocity. Only the value $c_1 > 0$ corresponds to the solution of the boundary layer, as follows from the first equation.

c) In the case $c_4 = 0$, $c_1 = -2c_3$, $c_2 = 2m$, $c_3 = 1$, we have

$$U\sim\frac{x}{t}m,\quad H_e\sim\frac{x}{t},\quad T_e\sim\frac{x^2}{t^2},\quad w^2\sim t.$$

The flow is described by the equations

$$\lambda G'''-2m(f''G-G''f)=-2G'-G''\eta.$$
$$f'''+f''\eta-2m(f'^2-ff'')+$$
$$+2m(1-S^2)-2(1-f')-$$
$$-2mS^2-(GG''-G'^2)=0,$$
$$P^{-1}\Theta''+2mf\Theta'-4m\Theta f'+$$
$$+\Theta'\eta+4\Theta+M^2(f''^2+S^2\lambda G''^2)=0.$$

d) In the case $c_1 = c_2 = 0$, $c_3 \neq 0$, $c_4 \neq 0$ we take $c_4 = \alpha$, $c_3 = 1-\alpha$, $0 \leq \alpha \leq 1$. Then the equations are reduced to the form

$$f'''+f''\{\eta+\alpha(f-\eta)\}-S^2GG''=0,$$
$$\lambda G'''+\alpha\{G''f-f''G\}+(1-\alpha)G''\eta=0, \qquad (1.10)$$

$$f'\Theta-|\Theta'\{\eta+\alpha(f-\eta)\}=P^{-1}\Theta''+M^2(f''^2+S^2\lambda G''^2),$$
$$f'\to1,\quad G'\to1,\quad \Theta\to1 \text{ when } \eta\to\infty,$$
$$f=f'=G'=G=0,\quad \Theta=\text{const when } \eta=0. \qquad (1.11)$$

This case describes a nonstationary flow over a plate set in sudden motion at a constant velocity (the Rayleigh problem). The value $\alpha = 0$ corresponds to the initial motion of the plate and $\alpha \to 1$ approximately to the stationary mode. Thus, the parameter α characterizes a change in the form of the equations from linear (initial motion) to nonlinear (stationary mode). An examination of the case $R_m < R$, $H_s(0, H_s, 0)$ completely duplicates the foregoing. For example, for the Rayleigh problem we have

$$f''' + f''\{\alpha\eta + (1-\alpha)f\} + n(1-f') = 0 \quad (n = c^{-2}\,\delta H_2^2),$$

$$2f'\Theta - \Theta'\{\alpha\eta + (1-\alpha)f\} =$$

$$= P^{-1}\Theta'' + M^2\{f''^2 + nf'(1-f')\}.$$

Since $H_2 = wH_S$,

$$H_s \sim \left[\alpha\frac{x}{U} + (1-\alpha)t\right]^{-1/2}.$$

The boundary conditions for the system are of the form

$$f' \to 1, \ \Theta \to 1 \text{ when } \eta \to \infty;$$

$$f = f' = 0, \ \Theta = \text{const when } \eta = 0.$$

2. In order to solve Eqs. (1.10), we make use of an iterative method. Weyl [10] has demonstrated the transformation of the Blasius differential equation to the integral equation

$$f''(\eta) = f''(0)\exp\left\{-\frac{1}{2}\int_0^\eta f''(u)(u-\eta)^2\,du\right\},$$

to which the iterative method, determined by the following conditions, is applied:

$$f''_{n+1} = \Psi\{f_n''\}.$$

This made it possible to obtain an analytic expression for f_1. Weyl showed that the sequence $\{f_n\}$ converges to a limiting function, which is a solution of the problem. Moreover,

$$f''(\eta) = \exp(-{}^1/_6\eta^2)$$

turns out to be an acceptable approximation to the limit. In reference [7], an iterative method is applied to solving the magnetic problem. For the case $\lambda = 1$, the convergence of the iterative process is proved there. This method is applied below for finding the solution of system (1.10).

A) Let us consider first the flow of a nonconducting fluid ($S = 0$). The equation of motion is of the form

$$f''' + f''\{\alpha f + (1-\alpha)\eta\} = 0, \qquad (2.1)$$

$$f' \to 1 \text{ when } \eta \to \infty, \qquad f = f' = 0 \text{ when } \eta = 0. \qquad (2.2)$$

We determine the iterative process in the following manner:

$$f_n''' + f_n\{\alpha f_{n-1} + (1-\alpha)\eta\} = 0,$$

$$f_n'(\infty) = 1, \ f_n(0) = f_n'(0) = 0.$$

This implies that

$$f_n''(\eta) = f_n''(0)\exp\left\{-\int_0^\eta \alpha f_{n-1}(u)\,du + (1-\alpha)\frac{\eta^2}{2}\right\},$$

$$\Phi_n = \Phi_n(0)\exp\left\{-\frac{\alpha}{2}\int_0^\eta \Phi_{n-1}(u-\eta)^2\,du + (1-\alpha)\frac{\eta^2}{2}\right\},$$

$$(\Phi_n \equiv f_n''). \qquad (2.3)$$

We select the initial function f_0 in the form

$$f_0 = c_0\eta, \qquad (2.4)$$

where c_0 is a still arbitrary constant. It can be seen from (2.3) that $\Phi_n(0)$ determines the friction on the wall $\eta = 0$. For a stationary flow, its value is found by means of numerical methods,

$$\Phi_0 = 0.4696. \qquad (2.5)$$

The arbitrariness in the selection of c_0 and the value of Φ_0 can be utilized to improve the accuracy of the approximation. We select c_0 so that when $\alpha = 1$

$$\Phi_1(0;1) = \Phi_0. \qquad (2.6)$$

It follows from (2.3) and (2.4) that

$$\Phi_1(\eta;\alpha) = \Phi_1(0;\alpha)\exp[-{}^1/_2(\alpha c_0 + 1 - \alpha)\eta^2]. \qquad (2.7)$$

From (2.6), (2.7), and the first condition of (2.2) we obtain

$$1 = \Phi_1(0;\alpha)\sqrt{{}^1/_2\pi/c_0} = \Phi_0\sqrt{{}^1/_2\pi/c_0}.$$

Hence

$$c_0 = {}^1/_2\pi\Phi_0^2 = 0.3469. \qquad (2.8)$$

For the nonstationary problem, we have from (2.7)

$$f'(\eta) = \sqrt{2}/\gamma\Phi_1(0;\alpha)\int_0^\eta e^{-\xi^2}d\xi \qquad (\gamma^2 = \alpha c_0 + 1 - \alpha). \qquad (2.9)$$

The value of $\Phi_1(0, \alpha)$ is determined here from the conditions

$$\Phi_1(0;\alpha) = \sqrt{2/\pi(\alpha c_0 + 1 - \alpha)} \quad \text{when } \eta \to \infty, \qquad (2.10)$$

$$\Phi_1(0;0) = \sqrt{2/\pi} = 0.71,$$

$$\Phi_1(0;1) = \sqrt{2c_0/\pi} = 0.47. \qquad (2.11)$$

A comparison of the expressions (2.11) shows that the friction in the initial phase of motion is almost twice as large as in the stationary case.

B) To solve the magnetohydrodynamic problem, we rewrite the equations (1.10) in the form

$$\Phi' + \Phi\{\eta + \alpha(f-\eta)\} - S^2G\chi = 0, \qquad f'' = \Phi,$$

$$\lambda\chi' + \alpha\{\chi f - G\Phi\} + (1-\alpha)\chi\eta = 0, \qquad G'' = \chi. \qquad (2.12)$$

We shall consider two sequences of functions f_n and G_n satisfying the equations

$$\Phi_n' + \Phi_n\{\eta + \alpha(f_{n-1} - \eta)\} - S^2G_{n-1}\chi_n = 0,$$

$$\lambda\chi_n' + \alpha\{\chi_n f_{n-1} - G_{n-1}\Phi_n\} + (1-\alpha)\chi_n\eta = 0,$$

and the boundary conditions

$$f_n(0) = f_n'(0) = G_n(0) = G_n'(0) = 0,$$

$$f_n'(\infty) = G_n'(\infty) = 1.$$

We select $f_0 = c_0\eta$, $G_0 = c_0\eta$ as initial conditions, where c_0 is in accordance with (2.8); then the equations for the first iterations are of the form

$$\Phi_1' + \Phi_1\{\alpha c_0 + (1-\alpha)\eta\} + S^2\alpha\eta c_0\chi_1 = 0,$$
$$\lambda\chi_1' + \alpha\{\chi_1 - \Phi_1\}c_0\eta + (1-\alpha)\chi_1\eta = 0.$$

By substituting the variable $\xi = \eta^2/2$, we reduce them to a system with constant coefficients:

$$\frac{d\mathbf{u}}{d\xi} + A\mathbf{u} = 0,\quad \mathbf{u} = \begin{pmatrix}\Phi_1\\ \chi_1\end{pmatrix},\quad A = \begin{pmatrix}\gamma^2 & -S^2\\ -\lambda^{-1} & -\gamma^2\lambda^{-1}\end{pmatrix}. \quad (2.13)$$

Here γ^2 is determined by (2.9). The solution of (2.13) is of the form

$$\Phi_1 = S^2\beta_1 e^{-m_1\xi} + S^2\beta_2 e^{-m_2\xi},$$
$$\chi_1 = \beta_1(\gamma^2 - m_1)e^{-m_1\xi} + \beta_2(\gamma^2 - m_2)e^{-m_2\xi}, \quad (2.14)$$

where m_1 and m_2 are the roots of the characteristic equation $A - mE = 0$

$$\lambda m_1 = {}^1/_2\gamma^2(1+\lambda) - \{\lambda S^2 + {}^1/_4\gamma^4(1-\lambda^2)\}^{1/2},$$
$$\lambda m_2 = {}^1/_2\gamma^2(1+\lambda) + \{\lambda S^2 + {}^1/_4\gamma^4(1-\lambda^2)\}^{1/2}.$$

Here m_2 is always positive and m_1 is positive when $0 \le S^2 \le \gamma^4$.

By integrating from zero to infinity, we obtain from (2.14)

$$f_1' = S^2\beta_1\int_0^\eta \exp\left(-\frac{m_1\eta^2}{2}\right)d\eta + S^2\beta_2\int_0^\eta \exp\left(-\frac{m_2\eta^2}{2}\right)d\eta,$$

$$G_1' = \beta_1(\gamma^2 - m_1)\int_0^\eta \exp\left(-\frac{m_1\eta^2}{2}\right)d\eta +$$
$$+ \beta_2(\gamma^2 - m_2)\int_0^\eta \exp\left(-\frac{m_2\eta^2}{2}\right)d\eta. \quad (2.15)$$

The constants β_1 and β_2 are determined from the equations

$$\frac{1}{S^2} = \beta_1\left(\frac{\pi}{2m_1}\right)^{1/2} + \beta_2\left(\frac{\pi}{2m_2}\right)^{1/2},$$
$$1 = \beta_1(\gamma^2 - m_1)\left(\frac{\pi}{2m_1}\right)^{1/2} + \beta_2(\gamma^2 - m_2)\left(\frac{\pi}{2m_2}\right)^{1/2},$$

and are respectively equal to

$$\beta_1 = \frac{S^2 + m_2 - 1}{S^2(m_2 - m_1)}\left(\frac{2m_1}{\pi}\right)^{1/2}, \qquad \beta_2 = -\frac{S^2 + m_1 - 1}{S^2(m_2 - m_1)}\left(\frac{2m_2}{\pi}\right)^{1/2}.$$

When $\eta = 0$, we obtain from (2.14)

$$\Phi(0;\alpha) =$$
$$= \frac{1}{m_2 - m_1}\left\{\left(\frac{\pi}{2m_2}\right)^{1/2}[S^2 + m_2 - 1] - \left(\frac{\pi}{2m_1}\right)^{1/2}[S^2 + m_1 - 1]\right\}.$$

The case $S \to 0$, $\lambda \to \infty$ corresponds to the motion of a nonconducting fluid. In this case, $m_2 \to \gamma^2$, $m_1 \to -0$, $S^2\beta_1 \to 0$, $S^2\beta_2 \to \gamma(2/\pi)^{1/2}$ and (2.14) becomes (2.10). For a fluid with high electric conductivity ($\lambda \to 0$)

$$m_1 \to \gamma^2, \qquad m_2 \to 0, \qquad S^2\beta_2 \to 0,$$
$$S^2\beta_1 \to \gamma^{-1}(1 - S^2)(2/\pi)^{1/2}.$$

It follows from (2.15) that in this case $G' \to 0$, that is, the magnetic field does not penetrate into the fluid. When $S^2 \to 1$, we have $f' \to 0$, $f'' \to 0$, and the flow is stopped by the magnetic field.

NOTATION

μ – viscosity coefficient, v – velocity, V – velocity of external flow, H – intensity of magnetic field, σ – conductivity of fluid, ν – coefficient of kinematic viscosity, R – magnetic Reynolds number, P – Prandtl number.

Primes denote differentiation; the subscript e denotes parameters of the external flow, and the subscript s wall parameters.

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